

Variance inequalities for quadratic forms with applications.

Pavel Yaskov¹

Abstract

We obtain variance inequalities for quadratic forms of weakly dependent random variables with bounded fourth moments. We also discuss two application. Namely, we use these inequalities for deriving the limiting spectral distribution of a random matrix and estimating the long-run variance of a stationary time series.

Keywords: Quadratic forms; Moment inequalities; Random matrices.

1 Introduction

Moment inequalities for quadratic forms constitute a powerful tool in time series analysis and the random matrix theory. In particular, they are used in the study of consistency and optimality properties of spectral density estimates (e.g., see Section V.4 in Hannan [9]) as well as provide low-level conditions under which the limiting spectral distribution of a random matrix can be derived (e.g., see Chapter 19 in Pastur and Shcherbina [19]).

When random variables $\{X_i\}_{i=1}^n$ are independent, moment inequalities for quadratic forms $\sum_{i,j=1}^n a_{ij} X_i X_j$ are well-studied (e.g., see Lemma B.26 in Bai and Silverstein [4] and Chen [5]). In the time series context, similar inequalities were obtained by many authors in connection with spectral density estimation and long-run variance estimation (e.g., see Chapter VI in White [21], Sections 6 and 7 in Wu and Xiao [22] and the references therein). In particular, high-order moment inequalities for causal time series were obtained by Wu and Xiao [22].

¹Steklov Mathematical Institute of RAS, Russia
National University of Science and Technology MISIS, Russia
e-mail: yaskov@mi.ras.ru
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In the present paper we study variance inequalities for quadratic forms $\sum_{i,j=1}^n a_{ij} X_i X_j$ of weakly dependent random variables $\{X_i\}_{i=1}^n$ with bounded fourth moments. Our assumptions deal with covariances of X_i 's products up to the fourth order only and are closely related to the classical fourth-order cumulant condition for a stationary time series (see Theorem V.4 in Hannan [9] and Assumption A in Andrews [1]). These assumptions can be easily verified under standard weak dependence conditions (e.g., strong mixing). We also demonstrate how our results can be applied in the random matrix theory and time series analysis.

The paper is structured as follows. Main results are given in Section 2. Section 3 is devoted to applications. Section 4 contains all proofs.

2 Main results

Let $\{X_k\}_{k=1}^\infty$ be a sequence of centred random variables and let $\{\varphi_k\}_{k=1}^\infty$ be a non-increasing sequence of non-negative numbers such that, for all $i < j < k < l$,

$$|\text{Cov}(X_i, X_j X_k X_l)| \leq \varphi_{j-i}, \quad |\text{Cov}(X_i X_j X_k, X_l)| \leq \varphi_{l-k}, \quad (1)$$

$$|\text{Cov}(X_i X_j, X_k X_l)| \leq \varphi_{k-j}, \quad \text{and} \quad |\text{Cov}(X_i, X_j)| \leq \varphi_{j-i}. \quad (2)$$

Assumptions of this kind go back to Révész [20] and studied by Komlós and Révész [13], Gaposhkin [7], and Longecker and Serfling [14] (see also Móricz [16], Section 4.3 and 4.4 in Doukhan et al. (2007)).

For simplicity, we will further assume that $\mathbb{E}X_k^2 \leq 1$, $k \geq 1$. Define $\mathbf{x}_p = (X_1, \dots, X_p)$ for $p \geq 1$, $\Phi_0 = \sup\{\mathbb{E}X_1^4, \mathbb{E}X_2^4, \dots\}$, and $\Phi_1 = \sum_{k=1}^\infty k\varphi_k$.

Theorem 2.1. *There is a universal constant $C > 0$ such that, for any $a \in \mathbb{R}^p$ and all $p \times p$*

matrices A with zero diagonal,

$$\mathbb{E}(\mathbf{x}_p^\top a)^4 \leq C(\Phi_0 + \Phi_1)(a^\top a)^2 \quad \text{and} \quad \text{Var}(\mathbf{x}_p^\top A \mathbf{x}_p) \leq C(\Phi_0 + \Phi_1)\text{tr}(AA^\top).$$

A version of the first inequality in Theorem 2.1 is proved by Komlós and Révész [13], Gaposhkin [7], and Longecker and Serfling [14]. The second inequality is new.

Let now ϕ_k , $k \geq 1$, satisfy

$$\text{Cov}(X_i^2, X_j^2) \leq \phi_{j-i} \quad \text{for all } i < j. \quad (3)$$

Define $\Phi_2 = \sum_{k=1}^{\infty} \phi_k$.

Theorem 2.2. *There is a universal constant $C > 0$ such that, for all $p \times p$ matrices A ,*

$$\text{Var}(\mathbf{x}_p^\top A \mathbf{x}_p) \leq C(\Phi_0 + \Phi_1 + \Phi_2)\text{tr}(AA^\top).$$

Let us give two examples of $\{X_k\}_{k=1}^{\infty}$ that satisfy (1), (2), and (3).

Example 1. Let $\{X_k\}_{k=1}^{\infty}$ be a martingale difference sequence with bounded 4th moments. Then (1) and (2) hold for $\varphi_k = 0$, $k \geq 1$, and $\Phi_1 = 0$. However, in general, there are no such ϕ_k , $k \geq 1$, that (3) holds and $\Phi_2 < \infty$. This explains why we introduce two sets of coefficients $\{\varphi_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$. If, in addition, $\{X_k^2 - \mathbb{E}X_k^2\}_{k=1}^{\infty}$ is a martingale difference sequence, then, of course, $\phi_k = 0$, $k \geq 1$, and $\Phi_2 = 0$.

Example 2. Let $\{X_k\}_{k=1}^{\infty}$ be strongly mixing random variables with mixing coefficients $(\alpha_k)_{k=1}^{\infty}$, zero mean, and bounded moments of order 4δ for some $\delta > 1$. Then (1)–(3) hold for $\varphi_k = \phi_k = C\alpha_k^{(\delta-1)/\delta}$ and large enough $C > 0$ (e.g., see Corollary A.2 in Hall and Heyde [8]). One can give similar bounds for other weak dependence conditions.

Remark 2.3. *We believe that higher order moment inequalities for quadratic forms $\mathbf{x}_p^\top A \mathbf{x}_p$ can be derived under similar conditions on $\text{Cov}(X_{i_1} \dots X_{i_k}, X_{i_{k+1}} \dots X_{i_p})$ for $i_1 < \dots < i_p$,*

$k = 1, \dots, p-1$, and $p > 4$. However, the proofs are quite technical even in the case of the second-order inequalities and we leave this question for future research.

Consider the special case when X_k , $k \geq 1$, are centred orthonormal random variables. In this case, (1) and (2) reduce to

$$|\mathbb{E}X_iX_jX_kX_l| \leq \min\{\varphi_{j-i}, \varphi_{k-j}, \varphi_{l-k}\}, \quad i < j < k < l.$$

Let $\mathbf{y}_p = (Y_1, \dots, Y_p)$, where each Y_j can be written as $\sum_{k=1}^{\infty} a_k X_k$ in L_2 for some $a_k \in \mathbb{R}$, $k \geq 1$, with $\sum_{k=1}^{\infty} a_k^2 < \infty$.

Corollary 2.4. *Let $\Sigma_p = \mathbb{E}\mathbf{y}_p\mathbf{y}_p^\top$. Then there is $C > 0$ such that, for any $p \times p$ matrix A ,*

$$\text{Var}(\mathbf{y}_p^\top A \mathbf{y}_p) \leq C(\Phi_0 + \Phi_1 + \Phi_2)\text{tr}(\Sigma_p A \Sigma_p A^\top).$$

If $\{X_k\}_{k=1}^{\infty}$ are independent standard normal variables and A is a $p \times p$ symmetric matrix, then $\text{Var}(\mathbf{y}_p^\top A \mathbf{y}_p) = 2\text{tr}((\Sigma_p A)^2)$ (e.g., see Lemma 2.3 in Magnus [15]). Thus, Corollary 2.4 delivers an optimal estimate of the variance up to the factor $C(\Phi_0 + \Phi_1 + \Phi_2)$.

3 Applications

In this section we discuss two applications of the obtained inequalities.

Our first application will be in the random matrix theory. For each $p, n \geq 1$, let \mathbf{Y}_{pn} be a $p \times n$ random matrix whose columns are independent copies of \mathbf{y}_p , where \mathbf{y}_p is given either in Corollary 2.4, or $\mathbf{y}_p = \mathbf{x}_p$ for \mathbf{x}_p from Theorem 2.2.

Theorem 3.1. *Let $\Phi_0, \Phi_1, \Phi_2 < \infty$. If the following conditions hold*

- (1) $p = p(n)$ is such that $p/n \rightarrow c$ for some $c > 0$,

(2) the spectral norm of $\Sigma_p = \mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top$ is bounded over p ,

(3) the empirical spectral distribution of Σ_p 's eigenvalues has a weak limit $P(d\lambda)$,

then, with probability one, the empirical spectral distribution of $n^{-1} \mathbf{Y}_{pn} \mathbf{Y}_{pn}^\top$'s eigenvalues has a weak limit whose Stieltjes transform $m = m(z)$ satisfies

$$m(z) = \int_0^\infty \frac{P(d\lambda)}{\lambda(1 - c - czm(z)) - z}, \quad z \in \mathbb{C}, \quad \text{Im}(z) > 0.$$

The next application concerns long-run variance estimation. First, let us recall the generic form of the central limit theorem for a weakly stationary time series $(X_t)_{t=-\infty}^\infty$:

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad n \rightarrow \infty,$$

where $\bar{X}_n = n^{-1} \sum_{t=1}^n X_t$, $\mathbb{E} X_t = \mu$, and σ^2 is the long-run variance of $(X_t)_{t=-\infty}^\infty$, i.e.

$$\sigma^2 = \sum_{j=-\infty}^\infty \text{Cov}(X_t, X_{t+j}).$$

This theorem can be proved under different weak dependence assumptions (e.g., see the books of Doukhan et al. [6] and Bulinski and Shashkin [4]). In statistical applications, this theorem takes the form

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n} \xrightarrow{d} \mathcal{N}(0, 1), \quad n \rightarrow \infty,$$

where $\hat{\sigma}_n^2$ is a consistent estimator of σ^2 . Recall also that σ^2 can be written as $\sigma^2 = 2\pi f(0)$, where $f = f(x)$, $x \in [-\pi, \pi)$, is the spectral density of $(X_t)_{t=-\infty}^\infty$. Therefore, long-run variance estimation is closely related to spectral density estimation.

A number of papers is devoted to the study of consistency and optimality properties of long-run variance estimators (e.g., see Andrews [1], Hansen [10], de Jong and Davidson

[12], and Jansson [11] among others). When $\mu = 0$, a typical estimator has the form

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{s,t=1}^n K\left(\frac{|s-t|}{m}\right) X_s X_t, \quad (4)$$

where $K = K(x)$, $x \geq 0$, is a kernel function. Standard assumptions on $K = K(x)$ include

- (a) $K(0) = 1$, K is continuous at $x = 0$, and $\sup_{x \geq 0} |K(x)| < \infty$,
- (b) $\int_0^\infty \bar{K}^2(x) dx < \infty$ for $\bar{K}(x) = \sup_{y \geq x} |K(y)|$,
- (c) $k_q = \lim_{x \rightarrow 0+} x^{-q}(K(x) - 1)$ exists for some $q > 0$.

Assumptions (a)–(b) are inspired by Assumption 3 of Jansson [11]. However, (b) is weaker than Assumption 3(ii) in [11], where the integrability of \bar{K} is assumed. To our knowledge, the weakest alternative to (b) considered in the literature is the integrability of K^2 . However, as discussed in Jansson [11], many previous results (Andrews [1], Hansen [10], among others) are incorrect as they stated and need stronger alternatives to the integrability of K^2 . Assumption (c) is classical and goes back to Parzen [18] (see also Andrews [1]).

Let further $(X_t)_{t=-\infty}^\infty$ be a centred weakly stationary time series that satisfies conditions from Section 2 (in particular, $\mathbb{E}X_t^2 \leq 1$). Our first result is the consistency of $\hat{\sigma}_n^2$.

Theorem 3.2. *Let $K = K(x)$ satisfy (a)–(b). If Φ_0, Φ_1, Φ_2 are finite, then $\hat{\sigma}_n^2 \rightarrow \sigma^2$ in mean square as $m, n \rightarrow \infty$ and $m = o(n)$.*

The dependence $m = o(n)$ is optimal. It can be seen by taking a Gaussian white noise $(X_t)_{t=-\infty}^\infty$ and showing that $\text{Var}(\hat{\sigma}^2) \not\rightarrow 0$ when $m/n \not\rightarrow 0$ due to the variance formula for Gaussian quadratic forms given in the end of Section 2. Andrews [1] following Hannan [9] proved consistency of $\hat{\sigma}_n^2$ under the cumulant condition

$$\sum_{j,k,l=1}^{\infty} \sup_{t \geq 1} |\kappa(X_t, X_{t+j}, X_{t+k}, X_{t+l})| < \infty. \quad (5)$$

Here $\kappa(X_i, X_j, X_k, X_l)$ is the fourth-order cumulant that is equal to

$$\mathbb{E}X_iX_jX_kX_l - \mathbb{E}X_iX_j\mathbb{E}X_kX_l - \mathbb{E}X_iX_k\mathbb{E}X_jX_l - \mathbb{E}X_iX_l\mathbb{E}X_k\mathbb{E}X_j$$

when each X_t has zero mean. By Lemma 1 of Andrews [1], (5) holds when $(X_t)_{t=-\infty}^{\infty}$ is a strongly mixing sequence with mixing coefficients $(\alpha_k)_{k=1}^{\infty}$ satisfying

$$\sum_{k=1}^{\infty} k^2 \alpha_k^{(\delta-1)/\delta} < \infty$$

and bounded moments of order 4δ for some $\delta > 1$. By Example 2, our Theorem 3.2 is applicable whenever $\sum_{k=1}^{\infty} k \alpha_k^{(\delta-1)/\delta} < \infty$.

The cumulant condition allows to calculate the limit of the mean squared error (MSE) of $\hat{\sigma}_n^2$ explicitly. We can not do it under our assumptions. However, we can give an upper bound for MSE which is very similar to the exact limit (see Proposition 1 in Andrews [1]).

Theorem 3.3. *Under conditions of Theorem 3.2, let (c) hold for some $q > 0$ and*

$$\text{the series } \Gamma_q = \sum_{j=1}^{\infty} j^q \text{Cov}(X_t, X_{t+j}) \text{ converges absolutely.}$$

Then there is an absolute constant $C > 0$ such that, as $m, n \rightarrow \infty$,

$$\mathbb{E}|\hat{\sigma}_n^2 - \sigma^2|^2 \leq C(\Phi_0 + \Phi_1 + \Phi_2) \frac{m}{n} \int_0^{\infty} \bar{K}^2(x) dx + \frac{4(k_q \Gamma_q)^2}{m^{2q}} + o(m^{-2q}) + O(n^{-1}).$$

4 Proofs

Below we assume that Φ_0, Φ_1, Φ_2 are finite otherwise all bounds become trivial.

Proof of Theorem 2.1. To prove the first inequality, we reproduce the proof given in Gaposhkin [7] with the only difference that we derive explicit constants in his inequality.

Note first that, as $\mathbb{E}X_i = 0$ for all $i \geq 1$, it follows from (1) that

$$|\mathbb{E}X_i X_j X_k X_l| \leq \min\{\varphi_{j-i}, \varphi_{l-k}\}, \quad i < j < k < l. \quad (6)$$

Write $a = (a_1, \dots, a_p)$. Using Lemma 1 in Moricz [16] with $p = 2$ and $r = 4$, we get

$$|(\mathbf{x}_p^\top a)^4 - 24T| \leq C_0(S^4 + |\mathbf{x}_p^\top a|^3 S),$$

where $C_0 > 0$ is a universal constant,

$$T = \sum_{i < j < k < l} a_i a_j a_k a_l X_i X_j X_k X_l, \quad S = \left(\sum_{i=1}^p a_i^2 X_i^2 \right)^{1/2},$$

hereinafter i, j, k, l are any numbers in $\{1, \dots, p\}$. By Hölder's inequality,

$$\mathbb{E}(\mathbf{x}_p^\top a)^4 \leq 24\mathbb{E}T + C_0(\mathbb{E}S^4 + (\mathbb{E}S^4)^{1/4}(\mathbb{E}|\mathbf{x}_p^\top a|^4)^{3/4}).$$

By (6),

$$|\mathbb{E}T| \leq \sum_{i < j < k < l} |a_i a_j a_k a_l| \min\{\varphi_{j-i}, \varphi_{l-k}\} \leq \frac{1}{4} \sum_{i < j < k < l} (a_i^2 + a_j^2)(a_k^2 + a_l^2) \min\{\varphi_{j-i}, \varphi_{l-k}\}.$$

We estimate only the term

$$J = \sum_{i < j < k < l} a_i^2 a_k^2 \min\{\varphi_{j-i}, \varphi_{l-k}\}.$$

Other terms with $a_j^2 a_k^2$, $a_i^2 a_l^2$, and $a_j^2 a_l^2$ instead of $a_i^2 a_k^2$ can be estimated similarly. We have

$$J \leq \sum_{i < k} a_i^2 a_k^2 \sum_{q, r=1}^{\infty} \min\{\varphi_q, \varphi_r\}$$

and

$$\sum_{q,r=1}^{\infty} \min\{\varphi_q, \varphi_r\} \leq \sum_{q=1}^{\infty} \left(q\varphi_q + \sum_{r=q+1}^{\infty} \varphi_r \right) = \Phi_1 + \sum_{r=2}^{\infty} \sum_{q=1}^{r-1} \varphi_r \leq 2\Phi_1. \quad (7)$$

As a result,

$$J \leq 2\Phi_1 \sum_{i < k} a_i^2 a_k^2 \leq \Phi_1 \left(\sum_{i=1}^p a_i^2 \right)^2 \quad \text{and} \quad |\mathbb{E}T| \leq \Phi_1 \|a\|^4.$$

Let us also note that

$$\mathbb{E}S^4 = \sum_{i,j=1}^p a_i^2 a_j^2 \mathbb{E}X_i^2 X_j^2 \leq \Phi_0 \|a\|^4.$$

Combining the above estimates, we infer that

$$\mathbb{E}(\mathbf{x}_p^\top a)^4 \leq (24 + C_0)(\Phi_0 + \Phi_1) \|a\|^4 + C_0 [(\Phi_0 + \Phi_1) \|a\|^4]^{1/4} [\mathbb{E}(\mathbf{x}_p^\top a)^4]^{3/4}.$$

Put $R = [\mathbb{E}(\mathbf{x}_p^\top a)^4 / (\Phi_0 + \Phi_1)]^{1/4} / \|a\|$. Then $R^4 \leq 24 + C_0 + C_0 R^3$. Therefore, $R \leq R_0$, where $R_0 > 0$ is the largest positive root of the equation $x^4 = 24 + C_0 + C_0 x^3$. Finally, we obtain

$$\mathbb{E}|\mathbf{x}_p^\top a|^4 \leq R_0^4 (\Phi_0 + \Phi_1) \|a\|^4.$$

We now verify the second inequality. First, note that, for $i < j$,

$$|\text{Cov}(X_i, X_j)| = |\mathbb{E}X_i X_j| \leq \sqrt{\mathbb{E}X_i^2 \mathbb{E}X_j^2} \leq 1$$

In addition, for $i < j < k < l$,

$$|\text{Cov}(X_i X_j, X_k X_l)| \leq 2 \min\{\varphi_{j-i}, \varphi_{k-j}, \varphi_{l-k}\} \quad \text{and} \quad I \leq 2 \min\{\varphi_{j-i}, \varphi_{l-k}\}, \quad (8)$$

where I is equal to $|\text{Cov}(X_i X_k, X_j X_l)|$ or $|\text{Cov}(X_i X_l, X_j X_k)|$. Indeed, by (2) and (6),

$$\begin{aligned} |\text{Cov}(X_i X_j, X_k X_l)| &\leq \min\{\varphi_{k-j}, |\mathbb{E}X_i X_j X_k X_l| + |\mathbb{E}X_i X_j \mathbb{E}X_k X_l|\} \\ &\leq \min\{\varphi_{k-j}, 2 \min\{\varphi_{j-i}, \varphi_{l-k}\}\} \\ &\leq 2 \min\{\varphi_{j-i}, \varphi_{k-j}, \varphi_{l-k}\}, \end{aligned}$$

and, by the monotonicity of φ_k ,

$$\begin{aligned} |\text{Cov}(X_i X_k, X_j X_l)| &\leq |\mathbb{E}X_i X_k \mathbb{E}X_j X_l| + |\mathbb{E}X_i X_j X_k X_l| \\ &\leq \min\{\varphi_{k-i}, \varphi_{l-j}\} + \min\{\varphi_{j-i}, \varphi_{l-k}\} \\ &\leq 2 \min\{\varphi_{j-i}, \varphi_{l-k}\}. \end{aligned}$$

A similar bound holds for $\text{Cov}(X_i X_l, X_j X_k)$.

Let $A = (a_{ij})_{i,j=1}^p$ and $a_{ii} = 0$, $1 \leq i \leq p$. Set $B = (A^\top + A)/2$. Then $\mathbf{x}_p^\top A \mathbf{x}_p = \mathbf{x}_p^\top B \mathbf{x}_p$ and

$$\text{tr}(BB^\top) = \sum_{i,j=1}^p \left(\frac{a_{ij} + a_{ji}}{2} \right)^2 \leq \sum_{i,j=1}^p \frac{a_{ij}^2 + a_{ji}^2}{2} = \sum_{i,j=1}^p a_{ij}^2 = \text{tr}(AA^\top). \quad (9)$$

Hence, we may assume w.l.o.g. that $A = A^\top$. Then

$$\begin{aligned} \text{Var}(\mathbf{x}_p^\top A \mathbf{x}_p) &= 4 \text{Var} \left(\sum_{i=1}^{p-1} X_i \sum_{k=i+1}^p a_{ik} X_k \right) = 4 \sum_{i=1}^{p-1} \text{Var} \left(X_i \sum_{k=i+1}^p a_{ik} X_k \right) + \\ &+ 8 \sum_{i < j} \text{Cov} \left(X_i \sum_{k=i+1}^p a_{ik} X_k, X_j \sum_{k=j+1}^p a_{jk} X_k \right) = 4I_1 + 8I_2 + 8I_3 + 8I_4, \end{aligned}$$

where

$$\begin{aligned}
I_1 &= \sum_{i=1}^{p-1} \text{Var} \left(X_i \sum_{k=i+1}^p a_{ik} X_k \right), \\
I_2 &= \sum_{i < j} \text{Cov} \left(X_i \sum_{k=i+1}^{j-1} a_{ik} X_k, X_j \sum_{k=j+1}^p a_{jk} X_k \right), \\
I_3 &= \sum_{i < j} a_{ij} \text{Cov} \left(X_i X_j, X_j \sum_{k=j+1}^p a_{jk} X_k \right), \\
I_4 &= \sum_{i < j} \text{Cov} \left(X_i \sum_{k=j+1}^p a_{ik} X_k, X_j \sum_{k=j+1}^p a_{jk} X_k \right),
\end{aligned}$$

and the sums over the empty set are zeros.

Control of I_1 . By the Cauchy-Schwartz inequality and the first inequality in Theorem 2.1,

$$\begin{aligned}
I_1 &\leq \sum_{i=1}^{p-1} \sqrt{\mathbb{E} X_i^4} \left(\mathbb{E} \left| \sum_{k=i+1}^p a_{ik} X_k \right|^4 \right)^{1/2} \\
&\leq C(\Phi_0 + \Phi_1) \sum_{i=1}^{p-1} \sum_{k=i+1}^p a_{ik}^2 = C(\Phi_0 + \Phi_1) \frac{\text{tr}(A^2)}{2}
\end{aligned}$$

Control of I_2 . By the Cauchy inequality and (8),

$$\begin{aligned}
I_2 &\leq \sum_{i < k < j < l} |a_{ik} a_{jl}| |\text{Cov}(X_i X_k, X_j X_l)| \\
&\leq 2 \sum_{i < k < j < l} |a_{ik} a_{jl}| \min\{\varphi_{k-i}, \varphi_{j-k}, \varphi_{l-j}\} \\
&\leq I_5 + I_6,
\end{aligned}$$

where

$$I_5 = \sum_{i < k < j < l} a_{ik}^2 \min\{\varphi_{j-k}, \varphi_{l-j}\}, \quad I_6 = \sum_{i < k < j < l} a_{jl}^2 \min\{\varphi_{k-i}, \varphi_{j-k}\}.$$

Additionally, by (7),

$$I_5 \leq \sum_{i < k} a_{ik}^2 \sum_{q,r=1}^{\infty} \min\{\varphi_q, \varphi_r\} \leq \frac{\text{tr}(A^2)}{2} (2\Phi_1) = \text{tr}(A^2)\Phi_1.$$

We similarly derive that $I_6 \leq \text{tr}(A^2)\Phi_1$. Hence, $I_2 \leq 2\text{tr}(A^2)\Phi_1$.

Control of I_3 . By the Cauchy-Schwartz inequality and the first inequality in Theorem 2.1,

$$\begin{aligned} I_3 &= \sum_{j=2}^{p-1} \text{Cov}\left(X_j \sum_{i=1}^{j-1} a_{ij} X_i, X_j \sum_{k=j+1}^p a_{jk} X_k\right) \\ &\leq \sum_{j=2}^{p-1} \left(\mathbb{E} X_j^2 \left| \sum_{i=1}^{j-1} a_{ij} X_i \right|^2\right)^{1/2} \left(\mathbb{E} X_j^2 \left| \sum_{k=j+1}^p a_{jk} X_k \right|^2\right)^{1/2} \\ &\leq \sum_{j=2}^{p-1} \sqrt{\mathbb{E} X_j^4} \left[\mathbb{E} \left(\sum_{i=1}^{j-1} a_{ij} X_i \right)^4 \mathbb{E} \left(\sum_{k=j+1}^p a_{jk} X_k \right)^4 \right]^{1/4} \\ &\leq \sqrt{C(\Phi_0 + \Phi_1)} (I_7 + I_8)/2, \end{aligned}$$

where

$$I_7 = \sum_{j=2}^{p-1} \left[\mathbb{E} \left(\sum_{i=1}^{j-1} a_{ij} X_i \right)^4 \right]^{1/2}, \quad I_8 = \sum_{j=2}^{p-1} \left[\mathbb{E} \left(\sum_{k=j+1}^p a_{jk} X_k \right)^4 \right]^{1/2}.$$

By the first inequality in Theorem 2.1,

$$I_7 \leq K \sum_{j=2}^{p-1} \sum_{i=1}^{j-1} a_{ij}^2 \leq \frac{K \text{tr}(A^2)}{2}, \quad I_8 \leq K \sum_{j=2}^{p-1} \sum_{k=j+1}^p a_{jk}^2 \leq \frac{K \text{tr}(A^2)}{2},$$

where $K = \sqrt{C(\Phi_0 + \Phi_1)}$. As a result, $I_3 \leq C(\Phi_0 + \Phi_1) \text{tr}(A^2)/2$.

Control of I_4 . We have $I_4 = I_9 + I_{10} + I_{11}$, where

$$I_9 = \sum_{i < j < k} \text{Cov}(a_{ik} X_i X_k, a_{jk} X_j X_k), \quad I_{10} = \sum_{i < j < k < l} a_{ik} a_{jl} \text{Cov}(X_i X_k, X_j X_l),$$

$$I_{11} = \sum_{i < j < k < l} a_{il} a_{jk} \text{Cov}(X_i X_l, X_j X_k).$$

By the first inequality in Theorem 2.1,

$$\begin{aligned} I_9 &= \frac{1}{2} \sum_{k=3}^p \text{Var} \left(X_k \sum_{i=1}^{k-1} a_{ik} X_i \right) - \frac{1}{2} \sum_{k=3}^p \sum_{i=1}^{k-1} \text{Var}(a_{ik} X_i X_k) \leq \\ &\leq \frac{1}{2} \sum_{k=3}^p \left[\mathbb{E} X_k^4 \mathbb{E} \left(\sum_{i=1}^{k-1} a_{ik} X_i \right)^4 \right]^{1/2} \leq C(\Phi_0 + \Phi_1) \sum_{k=3}^p \sum_{i=1}^{k-1} \frac{a_{ik}^2}{2} \leq C(\Phi_0 + \Phi_1) \frac{\text{tr}(A^2)}{4}. \end{aligned}$$

Let us now estimate I_{10} and I_{11} . By (8),

$$I_{10} \leq 2 \sum_{i < j < k < l} |a_{ik} a_{jl}| \min\{\varphi_{j-i}, \varphi_{l-k}\} \quad \text{and} \quad I_{11} \leq 2 \sum_{i < j < k < l} |a_{il} a_{jk}| \min\{\varphi_{j-i}, \varphi_{l-k}\}.$$

By the Cauchy inequality, $I_{10} \leq I_{12} + I_{13}$ and $I_{11} \leq I_{14} + I_{15}$ with

$$\begin{aligned} I_{12} &= \sum_{i < j < k < l} a_{ik}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\}, \quad I_{13} = \sum_{i < j < k < l} a_{jl}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\}, \\ I_{14} &= \sum_{i < j < k < l} a_{il}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\}, \quad I_{15} = \sum_{i < j < k < l} a_{jk}^2 \min\{\varphi_{j-i}, \varphi_{l-k}\}, \end{aligned}$$

As previously, we have

$$I_{12} \leq \sum_{i < k} a_{ik}^2 \sum_{q,r=1}^{\infty} \min\{\varphi_q, \varphi_r\} \leq \text{tr}(A^2) \Phi_1.$$

By the same arguments, I_{13} , I_{14} , and I_{15} can be bounded from above by $\text{tr}(A^2) \Phi_1$. Thus, we conclude that $I_{10} + I_{11} \leq 4 \text{tr}(A^2) \Phi_1$.

Combining all above estimates, we get $\text{Var}(\mathbf{x}_p^\top A \mathbf{x}_p) \leq C(\Phi_0 + \Phi_1) \text{tr}(A^2)$ for a universal constant $C > 0$. Q.e.d.

Proof of Theorem 2.2. Let $A = (a_{ij})_{i,j=1}^p$ and D be the $p \times p$ diagonal matrix with

diagonal entries a_{11}, \dots, a_{pp} . By Theorem 2.1,

$$\text{Var}(\mathbf{x}_p^\top (A - D)\mathbf{x}_p) \leq C(\Phi_0 + \Phi_1)\text{tr}((A - D)(A - D)^\top).$$

In addition, $\text{Var}(\mathbf{x}_p^\top A\mathbf{x}_p) \leq 2\text{Var}(\mathbf{x}_p^\top D\mathbf{x}_p) + 2\text{Var}(\mathbf{x}_p^\top (A - D)\mathbf{x}_p)$. Noting that

$$\text{tr}(AA^\top) = \text{tr}((A - D)(A - D)^\top) + \text{tr}(D^2),$$

we only need to bound $\text{Var}(\mathbf{x}_p^\top D\mathbf{x}_p)$ from above by $\text{tr}(D^2)$ up to a constant factor. Write $D = D_1 - D_2$, where D_i are diagonal matrices with non-negative diagonal entries and $\text{tr}(D^2) = \text{tr}(D_1^2) + \text{tr}(D_2^2)$. By the Cauchy inequality, $\text{Var}(\mathbf{x}_p^\top D\mathbf{x}_p) \leq 2 \sum_{i=1}^2 \text{Var}(\mathbf{x}_p^\top D_i\mathbf{x}_p)$. Hence, we may assume w.l.o.g. that diagonal elements of D are non-negative.

We see that

$$\begin{aligned} \text{Var}(\mathbf{x}_p^\top D\mathbf{x}_p) &= \text{Var}\left(\sum_{i=1}^p a_{ii}X_i^2\right) = \sum_{i=1}^p a_{ii}^2 \text{Var}(X_i^2) + \sum_{i \neq j} a_{ii}a_{jj} \text{Cov}(X_i^2, X_j^2) \\ &\leq \Phi_0 \sum_{i=1}^n a_{ii}^2 + \sum_{i \neq j} a_{ii}a_{jj}\phi_{|i-j|} \end{aligned}$$

and, as a result,

$$\begin{aligned} \text{Var}(\mathbf{x}_p^\top D\mathbf{x}_p) &\leq \Phi_0 \text{tr}(D^2) + \sum_{i \neq j} \frac{a_{ii}^2 + a_{jj}^2}{2} \phi_{|i-j|} \\ &\leq \Phi_0 \text{tr}(D^2) + \sum_{i=1}^p a_{ii}^2 \sum_{j: j \neq i} \phi_{|i-j|} \\ &\leq 2\text{tr}(D^2) \left(\Phi_0 + \sum_{k=1}^{\infty} \phi_k \right) = 2(\Phi_0 + \Phi_2)\text{tr}(D^2). \end{aligned}$$

Combining the above bounds, we get the desired inequality. Q.e.d.

Proof of Corollary 2.4. By the definition of \mathbf{y}_p , $\Gamma_n \mathbf{x}_n \rightarrow \mathbf{y}_p$ in probability and in mean

square as $n \rightarrow \infty$ for some $p \times n$ matrices Γ_n and $\mathbf{x}_n = (X_1, \dots, X_n)$. Since $X_k, k \geq 1$, are orthonormal, we have $\Gamma_n \Gamma_n^\top = \mathbb{E}(\Gamma_n \mathbf{x}_n)(\Gamma_n \mathbf{x}_n)^\top \rightarrow \mathbb{E} \mathbf{y}_p \mathbf{y}_p^\top = \Sigma_p$,

$$\mathbf{x}_n^\top (\Gamma_n^\top A \Gamma_n) \mathbf{x}_n = (\Gamma_n \mathbf{x}_n)^\top A (\Gamma_n \mathbf{x}_n) \rightarrow \mathbf{y}_p^\top A \mathbf{y}_p \quad \text{in probability,}$$

and

$$\mathbb{E} \mathbf{x}_n^\top (\Gamma_n^\top A \Gamma_n) \mathbf{x}_n = \text{tr}(\Gamma_n^\top A \Gamma_n) = \text{tr}(\Gamma_n \Gamma_n^\top A) \rightarrow \text{tr}(\Sigma_p A) = \mathbb{E} \mathbf{y}_p^\top A \mathbf{y}_p$$

as $n \rightarrow \infty$. We need the following version of Fatou's lemma:

$$\text{if } \xi_n \rightarrow \xi \text{ in probability, then } \mathbb{E}|\xi| \leq \varliminf_{n \rightarrow \infty} \mathbb{E}|\xi_n|.$$

By this lemma and Theorem 2.2,

$$\begin{aligned} \mathbb{E}|\mathbf{y}_p^\top A \mathbf{y}_p - \text{tr}(\Sigma_p A)|^2 &\leq \varliminf_{n \rightarrow \infty} \mathbb{E}|\mathbf{x}_n^\top (\Gamma_n^\top A \Gamma_n) \mathbf{x}_n - \text{tr}(\Gamma_n^\top A \Gamma_n)|^2 \\ &\leq \varliminf_{n \rightarrow \infty} C(\Phi_0 + \Phi_1 + \Phi_2) \text{tr}(\Gamma_n^\top A \Gamma_n \Gamma_n^\top A^\top \Gamma_n) \end{aligned}$$

Note that $\text{tr}(\Gamma_n^\top A \Gamma_n \Gamma_n^\top A^\top \Gamma_n) = \text{tr}(\Gamma_n \Gamma_n^\top A \Gamma_n \Gamma_n^\top A^\top) \rightarrow \text{tr}(\Sigma_p A \Sigma_p A^\top)$. Q.e.d.

Proof of Theorem 3.1. Denote the spectral norm of a matrix A by $\|A\|$. Recall that $\|A\| = \sqrt{\|AA^\top\|} = \sqrt{\|A^\top A\|}$. In addition, let $A^{1/2}$ be the principal square root of a square positive semi-definite matrix A . By Theorem 1.1 in Bai and Zhou [3], we will prove the theorem by checking that $\text{Var}(\mathbf{y}_p^\top A_p \mathbf{y}_p) = o(p^2)$ as $p \rightarrow \infty$ for any sequence $(A_p)_{p=1}^\infty$ with $\|A_p\| = O(1)$, where A_p is a $p \times p$ matrix.

First, let \mathbf{y}_p be as in Corollary 2.4. Then

$$\text{tr}(\Sigma_p A_p \Sigma_p A_p^\top) = \text{tr}(\Sigma_p^{1/2} A_p \Sigma_p A_p^\top \Sigma_p^{1/2}) = \text{tr}(Q \Sigma_p Q^\top)$$

with $Q = \Sigma_p^{1/2} A_p$. If I_p is the $p \times p$ identity matrix, then $\|\Sigma_p\| I_p - \Sigma_p$ is positive semi-definite

and, as a result, $Q(\|\Sigma_p\|I_p - \Sigma_p)Q^\top$ is positive semi-definite for any Q . Hence,

$$\text{tr}(Q^\top \Sigma_p Q) \leq \|\Sigma_p\| \text{tr}(QQ^\top) = \|\Sigma_p\| \text{tr}(Q^\top Q) = \|\Sigma_p\| \text{tr}(A_p^\top \Sigma_p A_p) \leq \|\Sigma_p\|^2 \text{tr}(A_p^\top A_p)$$

and $\text{tr}(A_p^\top A_p) \leq \|A_p\|^2 p$. Therefore, by Corollary 2.4,

$$\text{Var}(\mathbf{y}_p^\top A_p \mathbf{y}_p) \leq C(\Phi_0 + \Phi_1 + \Phi_2) \|A_p\|^2 \|\Sigma_p\|^2 p = o(p^2)$$

whenever $\|A_p\| = O(1)$. The case $\mathbf{y}_p = \mathbf{x}_p$ with \mathbf{x}_p as in Theorem 2.2 can be considered along the same lines due to the inequality $\text{tr}(A_p A_p^\top) \leq \|A_p\|^2 p$. Q.e.d.

Proof of Theorem 3.2. Since $\Phi_1 < \infty$, we have

$$\sum_{j=1}^{\infty} j|C(j)| < \infty \tag{10}$$

and σ^2 is well-defined, where $C(j) = \text{Cov}(X_t, X_{t+j})$, $j \in \mathbb{Z}$. We also have the bias-variance decomposition $\mathbb{E}(\hat{\sigma}^2 - \sigma^2)^2 = \text{Var}(\hat{\sigma}^2) + (\mathbb{E}\hat{\sigma}^2 - \sigma^2)^2$.

First, let us estimate the bias term $\mathbb{E}\hat{\sigma}^2 - \sigma^2$. Using $K(0) = 1$, we get

$$\begin{aligned} \mathbb{E}\hat{\sigma}^2 - \sigma^2 &= \sum_{j=-n}^n \left(1 - \frac{|j|}{n}\right) K\left(\frac{|j|}{m}\right) C(j) - \sum_{j=-\infty}^{\infty} C(j) \\ &= 2 \sum_{j=1}^n \left(K\left(\frac{j}{m}\right) - 1\right) C(j) - \frac{2}{n} \sum_{j=1}^n K\left(\frac{j}{m}\right) j C(j) - 2 \sum_{j>n} C(j) \end{aligned}$$

Now, setting $M = \sup_{x \geq 0} |K(x)|$,

$$\left| \sum_{j=1}^n K(j/m) j C(j) \right| \leq M \sum_{j=1}^{\infty} j |C(j)| = O(1).$$

Additionally,

$$\left| \sum_{j>n} C(j) \right| \leq \frac{1}{n} \sum_{j>n} j |C(j)| = o(1/n).$$

Combining these relations yields

$$\mathbb{E}\hat{\sigma}^2 - \sigma^2 = 2 \sum_{j=1}^n (K(j/m) - 1)C(j) + O(1/n) = o(1), \quad (11)$$

where the last equality follows from (a) and (10).

Now, consider the variance term $\text{Var}(\hat{\sigma}^2)$. By Theorem 2.2,

$$\text{Var}(\hat{\sigma}^2) \leq \frac{C_0}{n^2} \sum_{s,t=1}^n K^2\left(\frac{|s-t|}{m}\right),$$

where $C_0 = C(\Phi_0 + \Phi_1 + \Phi_2)$ with C given in Theorem 2.2. Using that $\bar{K}(x) = \sup_{y \geq x} |K(y)|$ is a non-decreasing function in $L_2(\mathbb{R})$, we derive

$$\frac{1}{mn} \sum_{s,t=1}^n K^2\left(\frac{|s-t|}{m}\right) \leq \frac{1}{m} + \frac{2}{m} \sum_{j=1}^n \bar{K}^2\left(\frac{j}{m}\right) \leq \frac{1}{m} + 2 \int_0^\infty \bar{K}^2(x) dx.$$

As a result,

$$\text{Var}(\hat{\sigma}^2) \leq \frac{C_0}{n} + \frac{2C_0m}{n} \int_0^\infty \bar{K}^2(x) dx \quad (12)$$

and $\text{Var}(\hat{\sigma}^2) = o(1)$ whenever $m, n \rightarrow \infty$ and $m/n \rightarrow 0$.

Combining the above bounds for the bias and the variance, we finish the proof. Q.e.d.

Proof of Theorem 3.3. The proof follows the same line as the proof of Theorem 3.2.

We only need to note the following. If (c) holds for some $q > 0$. Then, by (c) and the boundedness of K , $x^{-q}(K(x) - 1)$ is bounded on \mathbb{R}_+ . Therefore, by (a), (c), and the

absolute convergence of $\sum_{j \geq 1} j^q C(j)$,

$$m^q \sum_{j=1}^n (K(j/m) - 1) C(j) = \sum_{j=1}^n \frac{K(j/m) - 1}{(j/m)^q} j^q C(j) = k_q \sum_{j=1}^{\infty} j^q C(j) + o(1).$$

By (11) and (12), the latter yields the desired bound. Q.e.d.

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